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Scaling the ground water flow equation

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Abstract

Stochastic representation of aquifer parameters is adopted to incorporate their macroscopic variability (e.g. the horizontal variability at the scale of the order 1–10 m) into modeling the megascopic flow of ground water (i.e. the scale of the order of 1 km). The mathematical expression of the dynamics of megascopic ground water flow led to two deterministic coupled partial differential equations that must be solved simultaneously for the hydraulic head. The hydraulic head and the transmissivity are assumed to come from a joint Gaussian distribution, and the moment generating function is used to solve the closure problem. The assumption of a Gaussian distribution is more realistic for field applications than the commonly used perturbation techniques which neglect high-order moments of transmissivity and hydraulic head. The utility of the megascopic formulation of the ground water flow equation is demonstrated for the case of an aquifer in hydraulic connection with a stream. The small-scale macroscopic variability of aquifer transmissivity influences the megascopic behavior of the flow in the aquifer in both space and time. We propose to use the discrete kernels approach to reduce the amount of computations in stochastic ground water models.

1. Introduction

Research in ground water hydrology is needed at different ranges of descriptive scales. Haldorsen et al. (1988) identified four discrete scales of description: microscopic (pore and sand grains); macroscopic (core plug measurement scale); megascopic (field-scale simulation grid, layers); gigascopic (basin scale). Different scales of description were also recognized by Krause and Collins (1984), Dagan (1986), and

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Lasseter et al. (1986), among others. The understanding of the flow processes at a particular scale of description does not necessarily lead to an appropriate description at a larger scale. Indeed, the form of the governing equations may be entirely different, as in the passage from the Navier–Stokes equations at the pore level to Darcy's law at the laboratory scale (Ganoulis, 1989).

In the context of a continuum, fundamental to the passage from one discrete scale of description to the next, i.e. scaling the physical process, is the definition of a parameter that characterizes some property of the porous medium at the higher level of description. For example, in moving from the microscopic to the macroscopic scale of Darcy's law, Bear (1979) defined the representative elementary volume (REV) as the meaningful scale for describing porosity and hydraulic conductivity. He suggested that measuring devices should be designed to read values averaged over an REV of the porous medium. However, ground water hydrologists are interested in modeling ground water flow in aquifers of large areal extent. The descriptive scale in ground water models is not the REV, but some large operational megascopic scale (e.g. 1 km \times 1 km finite difference cell) that is dictated, to a large degree, by the limited number of spatially distributed measurements. The development of a formulation that permits the passage from the macroscopic to the megascopic scale of ground water flow models is the main objective of this paper.

The extrapolation of Darcy's law to a megascopic scale should, at the very least, account for the pronounced spatial variability at the macroscopic or measurement scale. Also, at this scale, the structure of this spatial variability, i.e. the relative position of the parameter values in space, does not follow a smooth pattern that is identifiable with a finite number of deterministic parameters. Because of limited resources to conduct field measurements, the areal extent of the aquifer over which the parameter is actually sampled is small compared with the modeled aquifer field. Consequently, the issue of extending or scaling the knowledge on spatial variability, beyond the measurements area, becomes important (e.g. Beven, 1993). The principles of analogy and induction, which constitute the essence of the stochastic theory, become tools to up-scale this knowledge on spatial variability. The geostatistical properties of the aquifer, i.e. the cumulative distribution function, the variance and covariance relationships, embedded in the data on transmissivity represent valuable information that can be extracted and incorporated into the new generation of ground water models.

A new formulation for the ground water flow equation which retains the fundamental physical aspect of the problem but allows for the spatial variability of transmissivity is developed. This formulation provides a set of deterministic coupled partial differential equation which can be solved in the same way as traditional ground water flow problems. Thus, the approach suggested can be viewed as a natural extension of the familiar deterministic modeling to account for known spatial variability at a scale smaller than the computational scale. The method is applied to an aquifer in hydrological connection with a stream. The objective was to demonstrate how macroscopic variability can influence the megascopic behavior of the aquifer both in space and time.

2. Theory

2.1. *The scale issue in existing ground water models*

The descriptive scale of aquifer transmissivity in the usual deterministic ground water models is the discrete computational scale, e.g. a constant value for the transmissivity inside a finite difference cell. To reveal a conceptual thorn in this classical approach, it is necessary to review the concept of a parameter. By definition, a parameter in a physical system, e.g. an aquifer, is a constant representative of a physical property of the system. In the context of a continuum, fundamental to the definition of a parameter is the presence of a scale in the range of description, over which the value of the property, e.g. aquifer transmissivity, converges to a meaningful invariant quantity. Is it justified physically to consider any computational scale a meaningful scale to specify an invariant property in it? In other words, can we define and calibrate parameters at any scale of discrete description? No. To fix the idea, let us superimpose a computational mesh on the relatively simple, but initially unknown, i.e. prior to calibration, heterogeneous field of Fig. 1. A single value that characterizes the heterogeneity inside the computational cell does not exist. The model that may struggle in the first place to calibrate some value for the average transmissivity inside the cell (calibrated under north–south flow conditions) is unaware of the fact that a different value for the average transmissivity is needed in the simulation mode (say east–west flow conditions). In fact, a different value for the effective transmissivity will prevail inside the computational cell for every temporal realization of the flow field. What is defined as a transmissivity parameter for the computational cell has an inherent stochastic nature, i.e. is non-unique. This is a problem in descriptive hydrology. The models do not use the measurement physical scale as the model scale, and, owing to small-scale heterogeneities, these scale-dependent parameters have an inherent stochastic nature. Morel-Seytoux and Nachabe (1992) argued that the same phenomenon occurs in modeling solute transport. The fact that the dispersion coefficient is, in practice, always scale dependent renders the Fickian description of the dispersion in an aquifer inapplicable. In ground water hydrology, three factors contribute to this stochastic scale effect: (1) the limited availability of measurements in the field; (2) the disparity between the measurement scale and the computational scale, the computational scale typically being orders of magnitude larger than the measurement scale; (3) the irregular variability at the macroscopic measurement scale, which defies a deterministic description. To capture this scale effect, a stochastic representation of the aquifer transmissivity is more realistic and should be adopted.

We propose to break the scale of description of the transmissivity into two components—a deterministic component to reflect the large-scale variability and a stochastic component to represent variabilities at a finer scale. In ground water flow models, the large-scale variability is accounted for by varying the transmissivity deterministically between the computational cells. Small-scale variabilities within a computational cell are accounted for statistically.

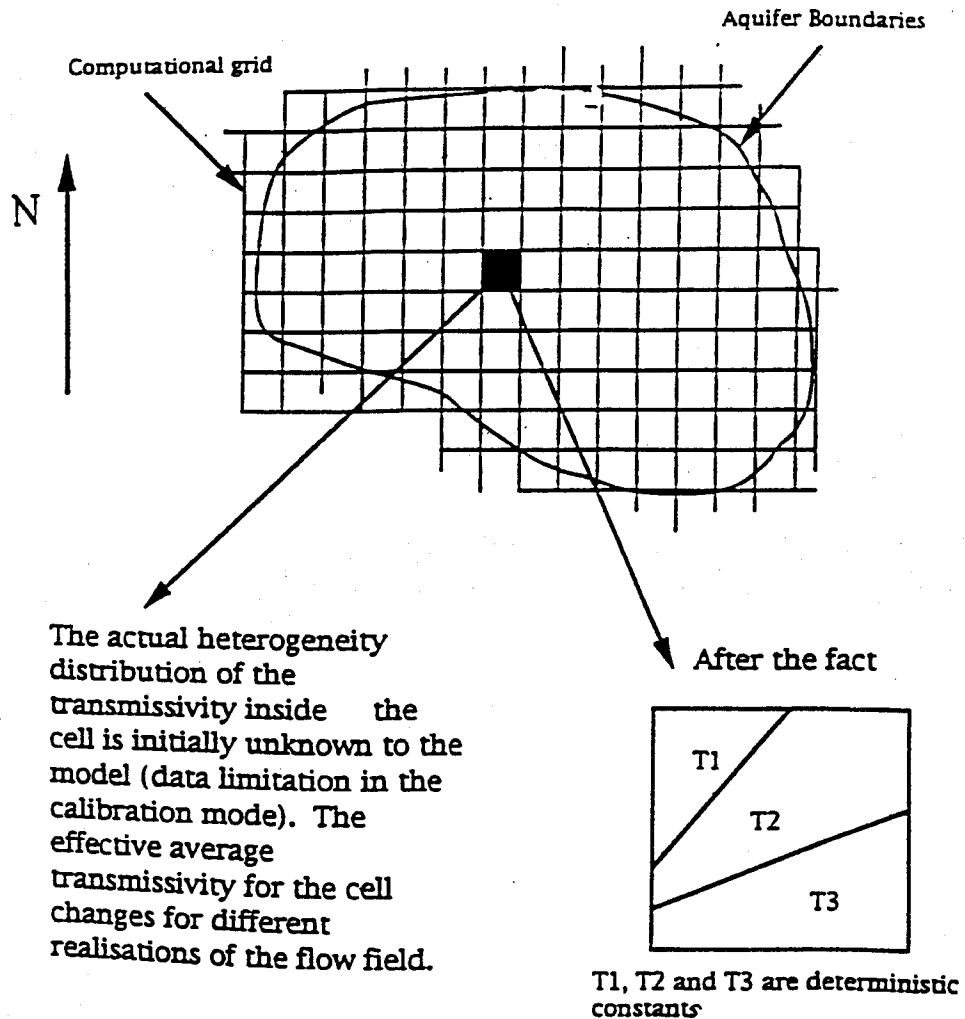


Fig. 1. The scale effect in ground water models. Can we define and calibrate parameters at any discrete scale of description?

2.2. Solution of stochastic flow problems and the Gaussian hypothesis

In the stochastic framework, the aquifer transmissivity is considered a realization of a log-normal spatial random function. Therefore, its joint frequency distribution function is multivariate normal, and is fully characterized by the first two moments: the mean, or the expected value of the transmissivity random variable, and the covariance function, a probabilistic measure of spatial correlation (e.g. Dagan, 1982a). By adopting this mathematical representation, the ground water flow equation becomes a stochastic differential equation with random coefficients.

Consequently, one may seek to determine the spatial probability distribution of the hydraulic head or, less ambitiously, the first two moments of the head random process.

A variety of procedures has been developed to solve stochastic flow problems. Dettinger and Wilson (1981) divided the various solution procedures into two main groups: full distribution analyses, and first and second moments analyses. Given the distribution functions of all the input parameters, the full distribution methods attempt to specify completely the probability distribution of the resulting hydraulic head process without introducing an assumption concerning the magnitude, i.e. the variance, of the random processes or the nature of the head probability distribution. A popular method to obtain the probability density functions is to use the Monte Carlo approach, which requires the performance of numerous flow simulations, with the parameters in each simulation generated at random from their respective probability distributions. In ground water hydrology, the Monte Carlo approach was utilized by Warren and Price (1961), and Smith and Freeze (1979), among others. However, Dettinger and Wilson (1981) noted that discretization of the transmissivity random process reduces its variance and increases its correlation length. In general, Graham and McLaughlin (1988) suggested that the size of the computational cell should be small enough to insure adequate reproduction of the probabilistic properties of the aquifer.

Most first two moments solution methods, whether in the spectral domain or the actual physical domain, are adaptations of the perturbation theory. To obtain valid solutions, the variance of the random variables must be small. The spectral approach generates analytical solutions but cannot handle nonstationarity when prescribing boundary conditions on the flow field (Gelhar, 1986). Nonstationarity is also introduced when Bayesian estimation techniques are used to condition prior statistics on field measurements, as demonstrated by Delhomme (1979). In this case, nonstationarity is very desirable because conditional statistics improve the description of field conditions. Perturbation solutions in the physical domain were developed by Dagan (1982a), who used Green's function to achieve analytical solutions. Sagar (1978) adopted a Galerkin finite element approximation to develop a numerical solution scheme. The perturbation solution is a Neuman series expansion that generates a sequential system of partial differential equations. The solution of this system of equations, which is usually truncated to the first order (Dagan, 1982a), converges to the actual solution when the magnitude of the transmissivity random variable is small.

As opposed to the perturbation techniques discussed above, we propose in this paper a new solution method that assumes a joint Gaussian distribution between the hydraulic head and the logarithm of the transmissivity. Both field investigations and numerical simulations suggest that this assumption is more suitable for modeling ground water flow than perturbation techniques. Monte Carlo simulations performed by Smith and Freeze (1979) indicated that the hydraulic head solution of the stochastic flow equation passes the normality test for a large portion of the aquifer domain.

Field investigations conducted by Hoeksema and Kitanidis (1985) indicated that

the probabilistic distribution function of the logarithm of transmissivity is multivariate normal. Hence, the proposed solution method provides a more realistic description of the transmissivity field than perturbation methods which truncate the known high-order moments of the input random variable. Also, because most solution techniques assume that the information concerning the head random process is adequately provided by the first two moments, it is rational to assume that the head probabilistic function is jointly normally distributed with the transmissivity. Hence, as will be demonstrated in the following section, the truncation of high-order moments can be avoided by expressing them in terms of low-order moments. The moment generating function is a convenient tool to derive the relationships between these moments.

Another argument that favors the use of a joint Gaussian distribution between the head and transmissivity is that in solving the inverse problem, i.e. in calibrating the transmissivity random field based on observations of head and transmissivity, one needs a prior adoption of some type of joint distribution between the head and transmissivity random variables. The joint Gaussian distribution is the only distribution considered and used by researchers in this field (e.g. Kitanidis and Vomvoris, 1983; Dagan, 1985). Also, applications of conditional probability require the prior specification of the head probabilistic distribution. Delhomme (1979) and Dagan (1982a), among others, assumed that the hydraulic head is normally distributed to condition field statistics.

3. Mathematical methods

The governing ground water flow equation in two dimensions has the form

$$\nabla \cdot [(T_r \nabla H_r)] + N = S \frac{\partial H_r}{\partial t} \quad (1)$$

where $\nabla \cdot$ is the divergence operator with respect to a spatial coordinate vector \mathbf{x} , N is the aquifer recharge (volume per unit area per unit time), t is time, S (dimensionless) is the storage coefficient for an unconfined aquifer or the specific yield of a phreatic aquifer, T_r is the transmissivity, and H_r is the hydraulic head. The subscript r indicates that the variable is random. The only source of stochasticity in Eq. (1) is the transmissivity. Hoeksema and Kitanidis (1985), among others, suggested that the transmissivity is a log-normal spatial random function. Hence the logarithm of $T_r(\mathbf{x})$ is written as

$$\ln T_r(\mathbf{x}) = \overline{\ln T_r}(\mathbf{x}) + \theta(\mathbf{x}) \quad (2)$$

where $\overline{\ln T_r}(\mathbf{x})$ is the expected value of $\ln T_r$ and $\theta(\mathbf{x})$ is the stochastic component with an expected value of zero. If the expectation operator is defined by $E(\cdot)$ then, mathematically, $E[\theta(\mathbf{x})] = \overline{\theta(\mathbf{x})} = 0$. Inverting Eq. (2) for T_r yields

$$T_r(\mathbf{x}) = T_L(\mathbf{x}) e^{\theta(\mathbf{x})} \quad (3)$$

where $T_L(\mathbf{x})$ is the geometric mean defined as

$$T_L(\mathbf{x}) = \exp[\ln T_r(\mathbf{x})] \quad (4)$$

Similarly, $H_r(\mathbf{x})$ is partitioned into two components, $H(\mathbf{x})$, the expected value of $H_r(\mathbf{x})$, and $h(\mathbf{x})$, the stochastic element with an expected value of zero. Symbolically,

$$H_r(\mathbf{x}) = H(\mathbf{x}) + h(\mathbf{x}) \quad (5)$$

where

$$E[H_r(\mathbf{x})] = \overline{H_r(\mathbf{x})} = H(\mathbf{x})$$

and

$$E[h(\mathbf{x})] = \overline{h(\mathbf{x})} = 0$$

Hence, the variability of transmissivity in Eq. (3) is represented by two components, a deterministic component $T_L(\mathbf{x})$ that reflects the large-scale megascopic variability (e.g. variability in different computational cells in a ground water model), and a stochastic component $\theta(\mathbf{x})$ to account for macroscopic variability that is usually superimposed on the large-scale variability. Because the transmissivity is log-normally distributed, the stochastic component $\theta(\mathbf{x})$ is fully characterized by its nonstationary covariance function $C_{\theta\theta}(\mathbf{x}, \mathbf{x}')$, where \mathbf{x}' is, like \mathbf{x} , a spatial coordinate. If the head and the transmissivity are jointly normal, the remaining unknowns required to characterize fully their joint frequency distribution are the first two moments of the head, $H(\mathbf{x}, t)$ and $C_{hh}(\mathbf{x}, \mathbf{x}', t)$, and the cross covariance moment between the head and the transmissivity $C_{h\theta}(\mathbf{x}, \mathbf{x}', t)$. Hence, whenever high-order moments, i.e. expectations of products of more than two random variables, emerge in the formulation, the moment-generating function is used (see the Appendix) to replace them by their equivalents in terms of their first- and second-order moments.

Replacing T_r and H_r by their expressions in Eqs. (2) and (5) into Eq. (1), the ground water flow equation takes the form

$$\nabla \cdot (T_L e^{\theta} \nabla H) + \nabla \cdot (T_L e^{\theta} \nabla h) = S \frac{\partial H}{\partial t} + S \frac{\partial h}{\partial t} - N \quad (6)$$

The second term on the left-hand side of Eq. (6) is rewritten in the following fashion:

$$\nabla \cdot (T_L e^{\theta} \nabla h) = \nabla \cdot [T_L (\nabla h e^{\theta})]_{\mathbf{x}'=\mathbf{x}} \quad (7)$$

The argument \mathbf{x}' is set equal to \mathbf{x} after the gradient operation is carried out and before the divergence operator is applied. Substituting Eq. (7) into Eq. (6) and taking expectation (Beran, 1968; Blanc-Lapierre and Fortet, 1968), yields

$$\nabla \cdot (T_L e^{\theta} \nabla H) + \nabla \cdot [T_L (\overline{\nabla h e^{\theta}})]_{\mathbf{x}'=\mathbf{x}} = S \frac{\partial H}{\partial t} - N \quad (8)$$

If $\theta(\mathbf{x})$ and $h(\mathbf{x})$ are jointly multivariate normal, the expectations of the high-order moments are expressed linearly in terms of lower-order moments using the

moment-generating function (details in Appendix). The resulting equation takes the form

$$\nabla \cdot (T_c \nabla H) + \nabla \cdot [T_c (\nabla C_{h\theta'})_{x'=x}] = S \frac{\partial H}{\partial t} - N \quad (9)$$

where T_c is the arithmetic average of the transmissivity distribution given symbolically as

$$T_c = T_L \exp(\sigma_\theta^2/2) \quad (10)$$

Where σ_θ is the standard deviation of $\theta(x)$. Eq. (9) has two unknowns, H and $C_{h\theta'}$, in one partial differential equation. Another partial differential equation that involves the cross covariance moment is obtained by multiplying Eq. (6) by $\theta'(x')$, the random component of the transmissivity at location x' , and taking the variables that are dependent on x' and x'' inside the derivative with respect to x :

$$\nabla \cdot (T_L \theta' e^\theta \nabla H) + \nabla \cdot [T_L (\nabla e^{\theta''} \theta' h)_{x''=x}] = S \theta'' \frac{\partial H}{\partial t} + S \frac{\partial h \theta'}{\partial t} - N \theta' \quad (11)$$

Taking the expectation of Eq. (11) (Blanc Lapierre and Fortet, 1968, Chapter 4) yields after substituting for the high-order moments in terms of low-order moments, using the moment-generating function,

$$\nabla \cdot (T_c C_{\theta\theta'} \nabla H) + \nabla \cdot (T_c \nabla C_{h\theta'}) + \nabla \cdot [T_c C_{\theta\theta'} (\nabla C_{h\theta'})_{x'=x}] = S \frac{\partial C_{h\theta'}}{\partial t} \quad (12)$$

The system of deterministic equations, Eqs. (9) and (12), involves two unknowns: the expected value of the head and the cross covariance moment between the head and the transmissivity. These two equations are coupled and must be solved simultaneously.

4. Results, discussion and illustrations

As opposed to perturbation methods (e.g. Sagar, 1978; Dagan, 1982a; Gelhar, 1986; McLaughlin and Wood, 1988), the closure problem is solved here without disregarding any high-order moments. Nachabe and Morel-Seytoux (1993) have shown that by introducing, in addition, a perturbation to their system of equations and truncating products of more than two random variables, Eqs. (9) and (12) reduce to those used by McLaughlin and Wood (1988). Hence, the Gaussian formulation presented here is not equivalent to a perturbation method. Also, Nachabe and Morel-Seytoux (1993) have shown that it is possible to obtain an equivalence between the perturbation and Gaussian formulations if the flow is steady (see also Gutjahr (1984)).

The structure of Eqs. (9) and (12) remains linear with the Gaussian approximation. This last finding implies that our solution approach does not require more numerical computations than a perturbation analysis. Because the variability of moments of random variables is smoother than the variability of random variables, the formulation presented above can be more suitable for numerical applications than

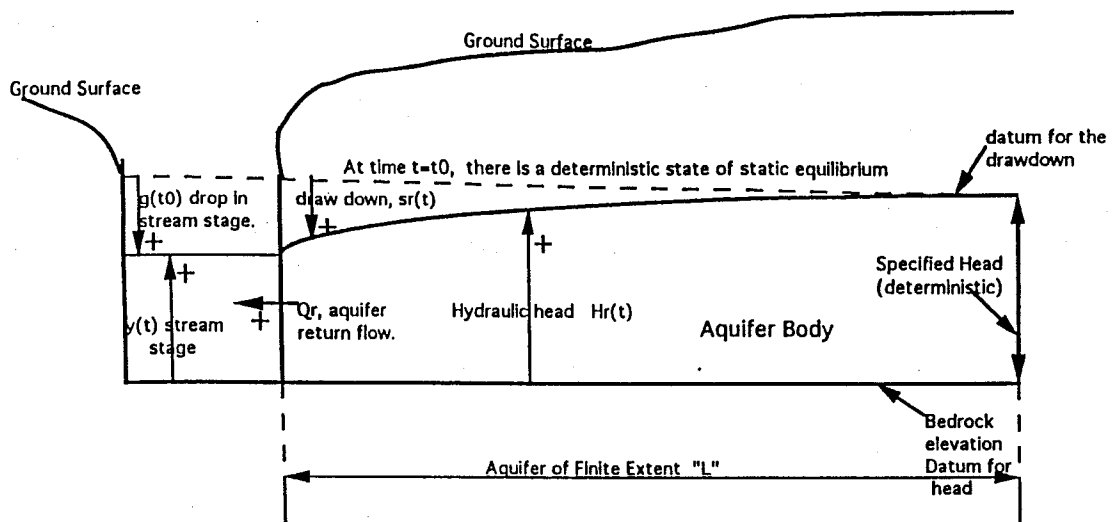


Fig. 2. The aquifer stream problem.

Monte Carlo simulations. Because the system of Eqs. (9) and (12) is linear and time invariant, the principle of superposition and the discrete kernels approach (e.g. Morel-Seytoux and Daly, 1975) can be used to solve ground water flow problems with time-dependent boundary conditions. This is important because, in many applications, the boundaries between the aquifer and the streams and lakes govern the distribution of water between the ground water and the surface water. Hence, quantifying the fluxes between the ground water and the surface water is essential for water resources management.

4.1. The stream–aquifer interaction problem

If the random component of the transmissivity, θ , is equal to zero, Eq. (9) becomes the classical deterministic ground water flow equation. Hence, macroscopic variability will influence the megascopic behavior of ground water flow. To demonstrate the role of this heterogeneity on ground water modeling, we consider the flow in an aquifer in hydrologic connection with a stream (see Fig. 2). The main objective is to quantify both the drawdown in the aquifer and the flux between the aquifer and the stream, i.e. the base flow or the aquifer return flow, owing to a pattern of fluctuation in the stage of the stream.

Initially, the aquifer, of finite extent L , is at rest and the hydraulic head is in equilibrium with the stage in the stream. At time t_0 , there is an instantaneous step drop of the stage in the stream, $g(t_0)$. Mathematically, the initial conditions on H is

$$H(x, t = t_0) = 0$$

The initial head distribution is assumed deterministic, i.e. $h(x, t = t_0)$ is identically

zero, and the initial condition on $C_{h\theta}$ is

$$C_{h\theta}(x, x', t = t_0) = 0$$

Also, because the random component of H_r is identically zero at the deterministic boundary, the conditions on H and $C_{h\theta}$ at $x = 0$ are

$$H(x = 0, t > 0) = y(t_0); \quad C_{h\theta}(x = 0, x', t > 0) = 0$$

For the sake of illustration, let the transmissivity T_r be a homogeneous process in both its mean, i.e. $T_c(x) = T_c = \text{constant}$, and its covariance function, i.e. $C_{\theta\theta}(x, x') = C_{\theta\theta}(x - x')$. The choice of the structure and parameters of the covariance function should be compatible with field observation (e.g. see Kitanidis and Vomvoris (1983) for the calibration of one-dimensional stochastic flow problems). For an exponential covariance model, the following expression is used:

$$C_{\theta\theta}(x - x') = (\sigma_\theta^2) \exp(-|x - x'|/D)$$

where D is the correlation length.

A finite difference numerical scheme was implemented to convert the coupled set of moments differential equations into a system of linear algebraic equations. Because the variability of these moments is relatively smooth in space, Graham and McLaughlin (1988) and McLaughlin and Wood (1988) suggested that a grid spacing around one correlation scale provides adequate numerical results for a linear covariance function. In the following simulations results, the computational grid was always smaller than the correlation scale. Because the flow is highly transient at early stages, a discrete time step that increases with the time of simulation was used.

For the sake of illustration, aquifer parameters of values $S = 0.15$, $T_L = 150 \text{ m}^2 \text{ day}^{-1}$; $D = 150 \text{ m}$, and $\sigma_\theta = 1$ were chosen for the reference run. The step drop in the stream stage was 10 m. Fig. 3 shows the evolution of the drawdown with time at $x = 600 \text{ m}$ from the stream bank for the reference run and for two deterministic runs with effective transmissivities equal to the arithmetic and harmonic averages of the transmissivity statistical distribution. For a log-normal distribution, the arithmetic average is $T_a = T_L \exp(\sigma_\theta^2/2)$, equal to $247.3 \text{ m}^2 \text{ day}^{-1}$, whereas the harmonic average is $T_h = T_L \exp(-\sigma_\theta^2/2)$, equal to $91.0 \text{ m}^2 \text{ day}^{-1}$.

To interpret the behavior of the drawdown in Fig. 3, one needs to understand the relationship between the spatial variability of the transmissivity and the diffusivity of the aquifer. First, in the case of no random macroscopic variability, the constant diffusivity is the ratio of transmissivity to specific yield. Large values of diffusivity indicate faster response by the aquifer and vice versa, i.e. it takes less time for the drawdown to reach 3 m for an arithmetic average than for a harmonic average of transmissivity (see Fig. 3). For the case of heterogeneous aquifers in one dimension, the spatial variability will tend to increase flow resistance; however, the impact of this spatial variability is barely felt at early times and the drawdown response is closer to that with the arithmetic average. At later times, spatial variability will have a more dramatic impact on flow resistance, and the drawdown response with time resembles more the response with a harmonic average for the transmissivity. These results are consistent with the findings in Dagan (1982b), which showed, for the case of infinite

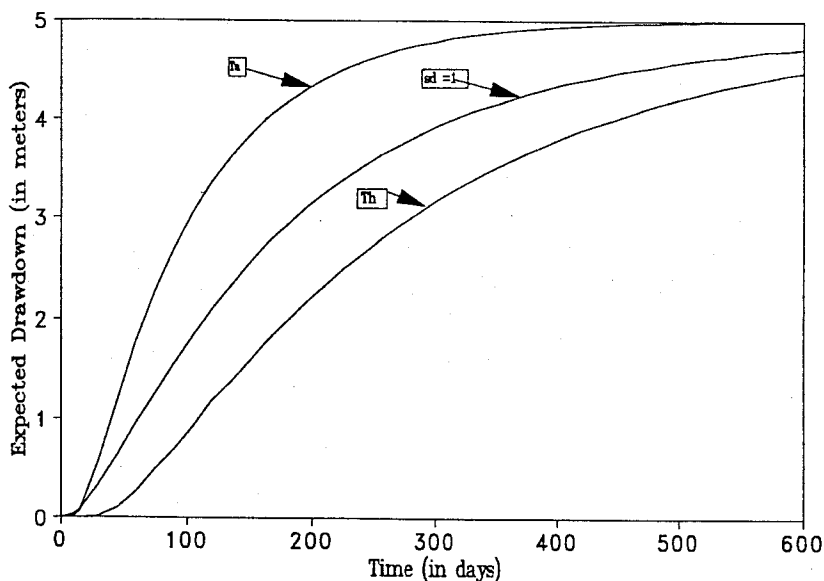


Fig. 3. Comparison of the drawdown evolution with time for the reference run (case $sd = 1$) and two deterministic simulations with the arithmetic average (labeled T_a) and harmonic (labeled T_h) averages of the transmissivity log-normal distribution.

spatial domain and slowly varying flow, that the time-dependent effective transmissivity relaxes gradually from an initial arithmetic average to a harmonic average at steady state. However, the comparison with Dagan's results is rather qualitative in this case, because, as pointed by Smith and Freeze (1979), the flow in a bounded domain may not exhibit an ergodic behavior. Indeed, Smith and Freeze (1979) emphasized that these nonergodic effects in spatial averages result in random fluxes for different realizations.

Fig. 4 compares the expected drawdown for three values of the standard deviation (sd) of the logarithm of transmissivity. The case $sd = 0$ corresponds to the geometric average of the transmissivity, given in Eq. (4). As expected, the evolution of the expected drawdown with time depends on the magnitude of $\theta(x)$.

Also, of major hydrological interest is the determination the total volume of water, $W(t)$, returned from the aquifer to the stream owing to the step drop in the stage of the stream. The expected cumulative water returned to the stream per unit length of the stream is given by the formula

$$W(t) = \int_{t_0}^t Q(x=0, \tau) d\tau$$

where Q is the flux at the interface between the aquifer and the stream, and τ is a dummy variable of integration with physical meaning of time. Fig. 5 shows the cumulative return volume from the aquifer to the stream for the reference run (case $sd = 1$) and for a deterministic run with an effective transmissivity equal to the

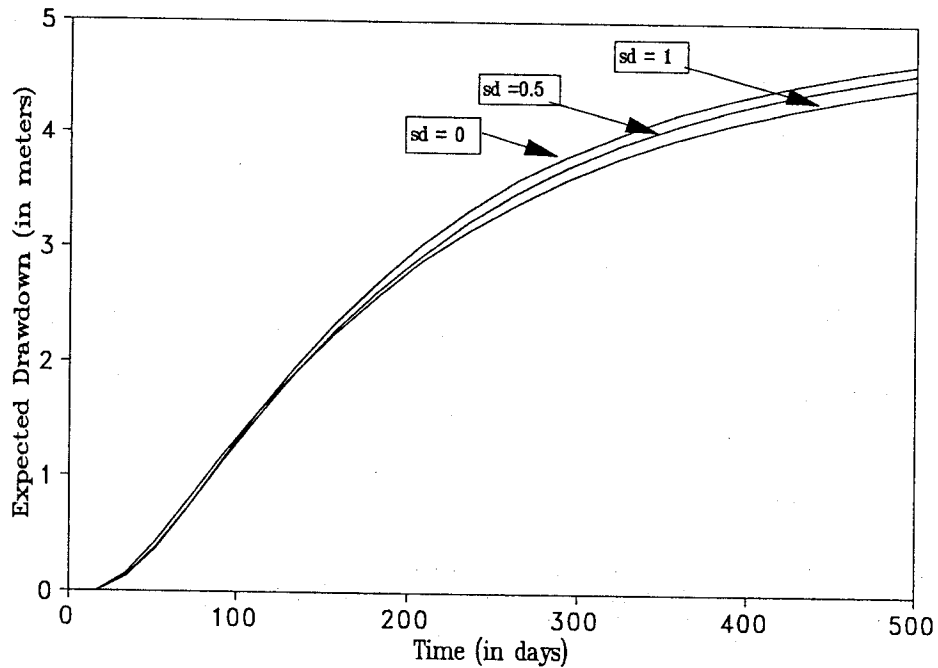


Fig. 4. The expected drawdown evolution with time for three cases of standard deviation (sd) of the logarithm of the transmissivity.

geometric mean (case $sd = 0$). For times less than 300 days from the beginning of the simulation, the aquifer yielded less water to the stream when the geometric mean of the transmissivity frequency distribution was used.

The results presented above are for a step drop in the stage in the stream. However, because the system of Eqs. (9) and (12) is linear and time invariant, the aquifer return flow owing to a succession of pulses of drop in the stream stage can be determined by superposition (Morel-Seytoux and Daly, 1975; Morel-Seytoux, 1991). For the monthly fluctuation pattern of the stage in the stream (see Table 1), we used the unit hydrograph formula and the discrete kernels approach (Morel-Seytoux and Daly, 1975) to tabulate the aquifer return flow for a period of 6 months.

5. Conclusion

A new formulation for ground water modeling that accounts for field heterogeneity at the macroscopic and megascopic scales of description was presented. It is widely perceived that the major practical benefit from a stochastic representation of hydrological variables is the ability to estimate prediction errors. Even though this issue is a growing concern among hydrologists and can be adequately addressed in a stochastic framework (McLaughlin and Wood, 1988), we believe in more compelling reasons to

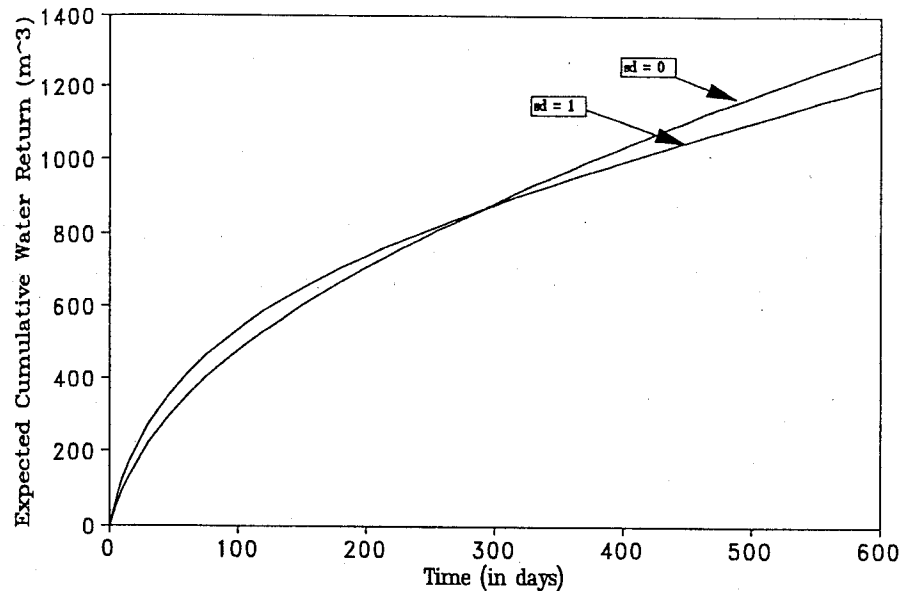


Fig. 5. Influence of heterogeneity on the cumulative water returned from the aquifer to the stream.

cast ground water modeling in a stochastic framework. Small-scale variability affects the megascopic interpretation of aquifer behavior in space and time. The stochastic scale effect is inherent in models, and we have listed three factors that enhance this scale effect: (1) the limited data available to the hydrologist; (2) the disparity between the model and the measurement scales; (3) the irregular variability of aquifer parameters. Stochastic representation should increase the utility of field measurements which assures better resemblance between model predictions and field conditions.

A promising solution method based on the Gaussian formulation was initiated. Although the computational burden in this new formulation is comparable with that for a perturbation analysis, the Gaussian formulation retains all higher-order moments. The Gaussian hypothesis which assumes that the first two moments of a frequency distribution exhaust the statistical properties of this distribution is more practical for field applications than perturbation solutions. Indeed, owing to scarcity of measurements, high-order moments are poorly quantified in the field, and most ground water modelers determine these first two moments only. Also, the Gaussian formulation is consistent with some Bayesian estimation techniques and with current methods to solve the inverse problem which require the prior specification of joint frequency distributions.

Table 1
Expected aquifer return flow for a variable pattern of stream stage

Time in periods (1 period = 30 days)	1	2	3	4	5	6
Stream stage drop (m)	2	6	5	2	-1	-4
Expected return flow (m ³ per period)	54.6	137.0	47.9	-41.7	-90.5	-124.4

Future applications of stochastic ground water models should consider the computational burden of the solution. The moments equations developed here are linear and time invariant. Hence, the discrete kernels approach is a rigorous and efficient procedure to reduce the amount of numerical computations. This approach should play a major role in the development of realistic solutions to stochastic flow problems in the future.

Appendix

The objective of this Appendix is to determine the expectations of higher-order moments of random variables. Under the multivariate normality assumption (both h and θ have an expectation of zero and are jointly normal), the moment-generating function (MGF for simplicity) is used to express the higher-order moments in terms of lower-order ones. By definition, the MGF, $M(u)$, for a single random variable z is given by the exponential transform (Haugen, 1968)

$$M(u) = \int_{\Omega} (e^{uz})f(z)dz \quad (\text{A1})$$

where $f(z)$ is the probability density function of the random variable z and the integration in Eq. (A1) is carried over the ensemble space Ω of the random variable z . Various moments of z are then obtained from Eq. (A1) by differentiation with respect to u . For example, the k th moment of z , $E(z^k)$, is simply

$$E(z^k) = \frac{\partial^k M}{\partial u^k} \Big|_{u=0} \quad (\text{A2})$$

If $f(z)$ is the Gaussian frequency distribution function, the integration in Eq. (A1) yields the MGF for the normally distributed random variable z . This MGF has the form

$$M(u) = \exp(\sigma_z^2/2) \quad (\text{A3})$$

This result is generalized to a vector z of order n , $\mathbf{z} = (z_1, z_2, \dots, z_n)$, of joint multivariate normal random variables. Haugen (1968) provided the following expression for the MGF, M_n :

$$M_n(u_1, u_2, \dots, u_n) = \exp\left(\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} u_i u_j\right) \quad (\text{A4})$$

where $\{\sigma_{ij}\}$ is the covariance matrix, a square matrix whose elements are the second moments of the variables. Symbolically, $\{\sigma_{ij}\}$ has the form

$$\begin{aligned} \sigma_{ij} &= E(z_i z_j) \\ \sigma_{i,i} &= \sigma_i^2 = E(z_i z_i) \end{aligned} \quad (\text{A5})$$

Several applications of the MGF are performed to determine the expectations of products of normally distributed random variables.

Case of univariate normal distribution: evaluation of $E(e^z)$

If the higher-order moment of interest is an exponential function of the random variable z , then it follows from the exponential transformation in Eq. (A1), and the result in Eq. (A3), that

$$E(e^z) = M(u = 1) = \left\{ \exp\left(\frac{1}{2}\sigma_{zz}u^2\right) \right\}_{u=1} = \exp\left(\frac{1}{2}\sigma_{zz}\right) \quad (\text{A6})$$

Case of bivariate normal distribution

The moment-generating function for the case of the bivariate normally distributed random variables z_1 and z_2 is obtained from Eq. (A4) by setting $n = 2$. The result is

$$M_2(u_1, u_2) = \exp\left\{\frac{1}{2}\sigma_{1,1}u_1^2 + \frac{1}{2}\sigma_{2,2}u_2^2 + \sigma_{1,2}u_1u_2\right\} \quad (\text{A7})$$

Then it follows from the definition of the MGF that

$$E[z_1 \exp(z_2)] = \left. \frac{\partial M_2}{\partial u_1} \right|_{u_1=0, u_2=1} \quad (\text{A8})$$

Substituting the expression in (A7) into (A8) yields

$$E[z_1 \exp(z_2)] = \sigma_{1,2} \exp(\sigma_2^2/2) \quad (\text{A9})$$

Case of trivariate normal distribution

The MGF for three jointly distributed normal random variables is obtained from Eq. (A4) by setting $n = 3$:

$$M_3(u_1, u_2, u_3) = \exp\left[\frac{1}{2}(\sigma_1^2u_1^2 + \sigma_2^2u_2^2 + \sigma_3^2u_3^2) + \sigma_{1,2}u_1u_2 + \sigma_{1,3}u_1u_3 + \sigma_{2,3}u_2u_3\right] \quad (\text{A10})$$

The moment $E[z_1z_2 \exp(z_3)]$ is determined from the relationship

$$E[z_1z_2 \exp(z_3)] = \left. \frac{\partial^2 M_3}{\partial u_1 \partial u_2} \right|_{u_1=0, u_2=0, u_3=1} \quad (\text{A11})$$

Substituting Eq. (A10) into Eq. (A11) yields the result

$$E[z_1z_2 \exp(z_3)] = (\sigma_{1,2} + \sigma_{1,3}\sigma_{2,3}) \exp(\sigma_3^2/2) \quad (\text{A12})$$

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References

- Bear, J., 1979. *Hydraulics of Groundwater*. McGraw-Hill, New York.
- Beran, M., 1968. *Statistical Continuum Theories*. Wiley, New York.
- Beven, K.J., 1993. Estimating transport parameters at the grid scale: on the value of a single measurement. *J. Hydrol.*, 143: 109-123.
- Blanc-Lapierre, A. and Fortet, R., 1968. *Theory of Random Functions, Vol. I*. Gordon and Breach, New York. (Translation from French by J. Gani.)
- Dagan, G., 1982a. Stochastic modeling of groundwater flow by unconditional and conditional probabilities, 1. Conditional simulation and the direct problem. *Water Resour. Res.*, 18(4): 813-833.
- Dagan, G., 1982b. Analysis of flow through heterogeneous random aquifers 2. Unsteady flow in confined aquifers. *Water Resour. Res.*, 18(5): 1571-1585.
- Dagan, G., 1985. Stochastic modeling of groundwater flow by unconditional and conditional probabilities: the inverse problem. *Water Resour. Res.*, 21(1): 65-79.
- Dagan, G., 1986. Statistical theory of groundwater flow and transport: pore to laboratory, laboratory to formations, and formations to regional scale. *Water Resour. Res.*, 22: 120S-135S.
- Delhomme, J., 1979. Spatial variability and uncertainty in ground water flow parameters: a geostatistical approach. *Water Resour. Res.*, 15(2): 269-280.
- Dettinger, M. and Wilson, J., 1981. First-order analysis of uncertainty in numerical models of groundwater flow, 1. Mathematical development. *Water Resour. Res.*, 17(1): 149-161.
- Ganoulis, J.G., 1989. Multiphase flow in porous media: description at the pore and macroscopic scale. In: H.J. Morel-Seytoux (Editor), *Unsaturated Flow in Hydrologic Modeling*. Kluwer Academic, Dordrecht, pp. 27-52.
- Gelhar, L., 1986. Stochastic subsurface hydrology from theory to applications. *Water Resour. Res.*, 22(9): 135S-145S.
- Graham, W. and McLaughlin, D., 1988. A comparison of numerical solution techniques for the stochastic analysis of nonstationary transient subsurface mass transport. In: *Proc. 7th International Conference on Computational Methods in Water Resources*. Computational Mechanics Publishers, Southampton, UK.
- Gutjahr, A., 1984. Stochastic models of subsurface flow: log linearized Gaussian models are exact for covariances. *Water Resour. Res.*, 20: 1909-1912.
- Haldorsen, H.H., Brand, P.J. and Macdonald, C.J., 1988. In: S. Edwards and P.R. King (Editors), *Review of the Stochastic Nature of Reservoirs, Mathematics in Oil Production*. Clarendon Press, Oxford, pp. 109-209.
- Haugen E., 1968. *Probabilistic Approaches to Design*. Wiley, New York.
- Hoeksema, R. and Kitanidis, P., 1985. Analysis of the spatial structure of properties of selected aquifers. *Water Resour. Res.*, 21(4): 563-572.
- Kitanidis, P. and Vomvoris, E., 1983. A geostatistical approach to the inverse problem in ground water modeling (steady state) and one dimensional simulations. *Water Resour. Res.*, 19(3): 677-690.
- Krause, F.F. and Collins, H.N., 1984. *Prembina cardium: Recovery Efficiency Study, a Geological and Engineering Synthesis*. Petroleum Recovery Institute, Calgary, Alta.
- Lasseter, T.J., Waggoner, J.R. and Lake, L.W., 1986. Reservoir heterogeneities and their influence on

- ultimate recovery. In: L.W. Lake and H.B. Carroll (Editors), Reservoir Characterization. Academic Press, San Diego, CA, pp. 545-559.
- McLaughlin, D. and Wood, E., 1988. A distributed parameters approach for evaluating the accuracy of ground water models predictions, 2. Application to groundwater flow. *Water Resour. Res.*, 24(7): 1048-1060.
- Morel-Seytoux, H.J., 1991. Analytical techniques for the solutions of ground water problems. Proc. 11th Annual AGU Hydrology Days. Hydrology Days Publ., Atherton, CA, pp. 217-260.
- Morel-Seytoux, H.J. and Daly, C., 1975. A discrete kernel generator for stream aquifer studies. *Water Resour. Res.*, 11(6): 968-972.
- Morel-Seytoux, H.J. and Nachabe, M., 1992. An effective scale-dependent dispersivity deduced from a purely convective flow field. *Hydrol. Sci. J.*, 37(2): 93-104.
- Nachabe, M.H. and Morel-Seytoux, H.J., 1993. On the application of the stochastic theory in ground water modeling. Proc. 13th Annual AGU Hydrology Days. Hydrology Days Publications, Atherton, CA, pp. 407-432.
- Sagar, B., 1978. Galerkin finite element procedure for analyzing flow through porous media. *Water Resour. Res.*, 14(6): 1035-1044.
- Smith, L. and Freeze, R.A., 1979. Stochastic analysis of steady ground water flow in a bounded domain, 1—One dimensional simulations. *Water Resour. Res.*, 15(3): 521-528.
- Warren, J. and Price, H., 1961. Flow in heterogeneous porous media. *Soc. Petrol. Eng. J.*, 1: 153-169.