

Analytical Solutions for Water Flow and Solute Transport in the Unsaturated Zone

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Abstract

Analytical solutions for the water flow and solute transport equations in the unsaturated zone are presented. We use the Broadbridge and White nonlinear model to solve the Richards' equation for vertical flow under a constant infiltration rate. Then we extend the water flow solution and develop an exact parametric solution for the advection-dispersion equation. The method of characteristics is adopted to determine the location of a solute front in the unsaturated zone. The dispersion component is incorporated into the final solution using a singular perturbation method. The formulation of the analytical solutions is simple, and a complete solution is generated without resorting to computationally demanding numerical schemes. Indeed, the simple analytical solutions can be used as tools to verify the accuracy of numerical models of water flow and solute transport. Comparison with a finite-element numerical solution indicates that a good match for the predicted water content is achieved when the mesh grid is one-fourth the capillary length scale of the porous medium. However, when numerically solving the solute transport equation at this level of discretization, numerical dispersion and spatial oscillations were significant.

Introduction

The movement of water and solutes through the unsaturated zone has been of importance in traditional applications of ground-water hydrology, soil physics, and agronomy. In recent years, the need to understand the behavior of hazardous waste and toxic chemicals in soils has resulted in a renewed interest in this subject. One of the primary concerns is that dissolved contaminants may migrate through the unsaturated zone, reach the saturated zone, and contaminate the ground water.

The search for analytical solutions to model water flow and solute transport continues to be of scientific interest. Typically, water flow and solute transport in unsaturated soils result in transient phenomena, making it a challenging problem. The nature of soil hydraulic properties like diffusivity and hydraulic conductivity renders the governing flow equation nonlinear. In recent years, several analytical methods were developed to simulate water movement and

solute transport in the unsaturated zone (e.g., Broadbridge and White, 1988; McWhorter and Sunada, 1990; Warrick et al., 1991; Islas and Illangasekare, 1992; Barry et al., 1993). In this paper, the Broadbridge and White (B&W, hereafter) nonlinear flow model for constant rate infiltration is adopted. The ability of this model to conform to a variety of soil types, as demonstrated by White and Broadbridge (1988) and White and Perroux (1989), is very encouraging in field applications. Gee et al. (1991) indicated that this is the first exact and realistic analytical solution for water flow under constant infiltration rate. The objective of this paper is to capitalize on these features of that analytical flow model and extend its use to simulate contaminant movement in soils so that a complete, closed form analytical solution for solute transport in the unsaturated zone is now achieved. The developed analytical solutions will provide the tools to test the numerical approximations in existing water flow and solute transport codes.

In the first two sections of this paper, the solute transport problem is stated mathematically. The convective part of the solute transport equation is determined using the method of characteristics. The dispersive component is incorporated into the solution using the method of singular perturbation in section 4. The analytical solutions are compared with a finite-element numerical solution.

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1. Assumptions and Mathematical Statement of the Problem

Two fundamental equations describe water flow and solute transport in unsaturated soils. Richards' equation, which describes water flow, is obtained by combining Darcy's law for unsaturated flow with the continuity equation. For one-dimensional vertical flow, this equation is:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[D(\theta) \frac{\partial \theta}{\partial z} \right] - \frac{\partial K(\theta)}{\partial z} \quad (1)$$

The transport of a conservative (i.e., nonvolatile, nonsorbing, nondegrading) solute in the unsaturated zone is described by the advection-dispersion equation written in the form:

$$\frac{\partial(\theta c)}{\partial t} = -q \frac{\partial c}{\partial z} + \frac{\partial}{\partial z} \left[\theta E \frac{\partial c}{\partial z} \right] \quad (2)$$

In both equations, t is time; z is depth taken zero at the soil surface and positive downward; $\theta(z, t)$ is volumetric soil water content; $D(\theta)$ is soil-water diffusivity; $K(\theta)$ is unsaturated soil hydraulic conductivity; $c(z, t)$ is solute concentration (mass of solute over volume of solution); $q(z, t)$ is Darcy's water flux given by the relationship:

$$q(z, t) = K(\theta) - D(\theta) \frac{\partial \theta}{\partial z} \quad (3)$$

and E is a hydrodynamic dispersion coefficient, a parameter that lumps the impacts of molecular diffusion and mechanical mixing. Equations (1), (2), and (3) are coupled and they should be solved sequentially. First, the flow equation is solved for $\theta(z, t)$ and the subsequent substitution of $\theta(z, t)$ into equation (3) provides the unsaturated Darcy's flux $q(z, t)$. The solutions $\theta(z, t)$ and $q(z, t)$ become input for equation (2). There are two basic contributions to the transport of a conservative solute in the unsaturated zone. The first term on the right side of equation (2) represents transport by advection with the mean pore-water velocity. This term is important when significant liquid flow (water flux q) is involved. The second term on the right-hand side of equation (2) accommodates for dispersion due to the spatial variation of the velocity around the mean. The dispersion solute flux is assumed Fickian and E , the hydrodynamic dispersion coefficient, includes molecular diffusion and mechanical dispersion (e.g., Bear, 1979). Intuitively, the molecular diffusion component plays a significant role in the transport of a conservative solute when minor water flow, i.e. small q , is present.

Equation (2) is reduced to a more practical mathematical form using the continuity equation for water flow:

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0 \quad (4)$$

Substitution of equation (4) into equation (2) yields:

$$\theta \frac{\partial c}{\partial t} = -q \frac{\partial c}{\partial z} + \frac{\partial}{\partial z} \left(\theta E \frac{\partial c}{\partial z} \right) \quad (5)$$

Equation (5) applies for transient transport of conservative solutes in unsaturated soils.

2. The Broadbridge and White Nonlinear Flow Model

In the analytical solution of the Richards' equation, B&W assumed specific boundary and initial conditions. At the soil surface a constant infiltration rate, q_0 , is prescribed as:

$$q = q_0, \quad z = 0, \quad t \geq 0 \quad (6)$$

Initial conditions corresponding to a uniform soil-water content in the soil horizon are specified as:

$$\theta(z, 0) = \theta_n, \quad z \geq 0 \quad (7)$$

Broadbridge and White (1988) adopted a functional form for the diffusivity given by Philip and Knight (1974) which allows for the transformation of the nonlinear Richards' equation to a linear form for which an analytical solution exists. This diffusivity function has the form:

$$D(\theta) = \frac{a}{(b - \theta)^2} \quad (8)$$

where a and b are constants. As a second step in the solution of the nonlinear flow problem, B&W (1988) developed an expression for $K(\theta)$ which in conjunction with the assumed function for $D(\theta)$ transforms equation (1) to the weakly nonlinear Burger's equation. This expression for $K(\theta)$ is given as:

$$K(\theta) = \beta + \gamma(b - \theta) + \frac{\lambda}{2(b - \theta)} \quad (9)$$

where β , γ , and λ are constants. With the suggested analytical forms for $K(\theta)$ and $D(\theta)$ and the imposed boundary conditions, the Kirchhoff-Sturm and the Hopf-Cole transformations are applied to reduce equation (1) to a linear form that possesses an exact parametric solution.

Then by introducing the parameter C ,

$$C = \frac{b - \theta_n}{\Delta \theta} \quad (10)$$

with $\Delta \theta = \theta_s - \theta_n$, and the relative water content θ_* ,

$$\theta_* = \frac{\theta - \theta_n}{\Delta \theta} \quad (11)$$

the hydraulic conductivity and diffusivity functional forms in equations (8) and (9) are written in the convenient forms (B&W, 1988):

$$K_*(\theta_*) = \frac{K - K_n}{\Delta K} = \frac{C - 1}{C - \theta_*} \theta_*^2 \quad (12)$$

$$D_*(\theta_*) = \frac{t_s}{\lambda_s^2} D(\theta) = \frac{C(C - 1)}{(C - \theta_*)^2} \quad (13)$$

The notation $(\cdot)_*$ implies that the variable is dimensionless. $\Delta K = K_s - K_n$ where $K_s = K(\theta_s)$ and $K_n = K(\theta_n)$, and λ_s is the capillary length scale, a K -weighted mean soil-water potential expressed as:

$$\lambda_s = (\Delta K)^{-1} \int_{\theta_n}^{\theta_s} D(\theta) d\theta \quad (14)$$

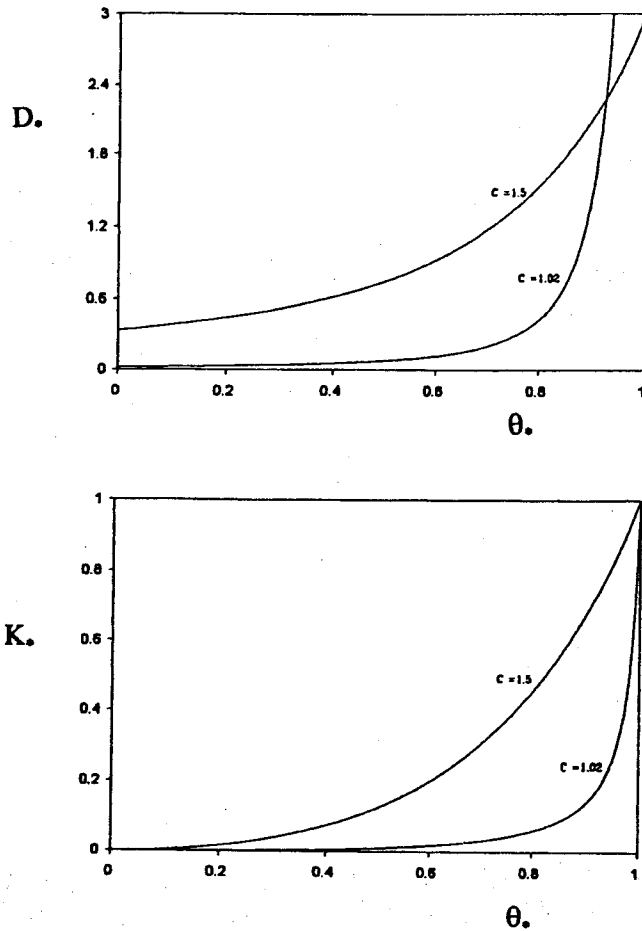


Fig. 1. Dimensionless diffusivity D_* and conductivity K_* versus normalized water content θ_* for two values of C .

It is recognized that λ_s could be thought of as the mean height of the capillary fringe above a water table. It is also equivalent to the inverse of the sorptive number of the quasi-linear hydraulic conductivity model (White and Sully, 1987; Nachabe and Illangasekare, 1994). The sorptive time scale, t_s in equation (13), emerges from the transient flow equation of B&W as (White and Sully, 1987):

$$t_s = \frac{\Delta\theta\lambda_s}{\Delta K} \quad (15)$$

Figure 1 shows $K_*(\theta)$ and $D_*(\theta)$ for two values of C . The relationship between the capillary pressure, ψ , and the moisture content, θ , is derived from (12) and (13) by using the definition of $D(\theta)$, namely, $D(\theta) = K(\theta) (d\psi/d\theta)$. The resulting equation is written as:

$$\psi_*(\theta_*) = -\frac{1-\theta_*}{\theta_*} - C^{-1} \log \frac{C-\theta_*}{\theta_*(C-1)} \quad (16)$$

where $\psi_* = \psi/\lambda_s$. Figure 2 shows $\psi_*(\theta_*)$ for several values of C . As C tends to 1, the capillary pressure shows a pronounced air entry pressure which is typical of laboratory packed materials or fine soils. As C increases, the "knee" in the normalized capillary pressure curve disappears, and equation (16) results in a water retention curve similar to those found in intact field soils (e.g., Perroux et al., 1982).

3. Analytical Solution for Richards' Equation

The water content $\theta(z, t)$ solution of equation (1) was determined by B&W (1988). It is presented here for the sake of completeness. The substitution for the expressions of K_* and D_* in equations (12) and (13) into the flow equation provides the dimensionless form of Richards' equation:

$$\frac{\partial\theta_*}{\partial t_*} = \frac{\partial}{\partial z_*} \left[D_*(\theta_*) \frac{\partial\theta_*}{\partial z_*} \right] - \frac{\partial K_*(\theta_*)}{\partial z_*} \quad (17)$$

where $z_* = z/\lambda_s$ and $t_* = t/t_s$. Similarly the initial and boundary conditions in equations (6) and (7) are expressed in dimensionless form as:

$$\theta_* = 0 \quad t_* = 0, z_* \geq 0 \quad (18)$$

$$K_*(\theta_*) - D_*(\theta_*) \frac{\partial\theta_*}{\partial z_*} = q_{*,0} \quad z_* = 0, t_* \geq 0 \quad (19)$$

where $q_{*,0} = (q_0 - K_n)/\Delta K$. The solution for equation (17) subject to the conditions in equations (18) and (19) is then given by (B&W, 1988):

$$\theta_* = C \left(1 - \frac{1}{2\rho + 1 - u^{-1} u_\zeta} \right) \quad (20)$$

$$z_* = C^{-1} [\rho^2(1 + \rho^{-1}) \tau + \rho(2 + \rho^{-1}) \zeta - \ln u] \quad (21)$$

The analytical expressions appearing in (20) and (21) are documented in the appendix. The main features of this solution are discussed in the work of B&W (1988), but it is emphasized that equations (20) and (21) constitute an exact parametric solution with parameter ζ .

4. Parametric Solution for the Advection-Dispersion Equation

We capitalize on the analytical form for the water flow solution and pursue an extension of this result to the movement of a conservative solute in the unsaturated zone. This is achieved by coupling the advection-dispersion equation with the water content solution in equations (20) and (21). Wilson and Gelhar (1981) developed analytical solutions for the advection-dispersion equation, but as opposed to the result presented here, their solution for water flow in the soil

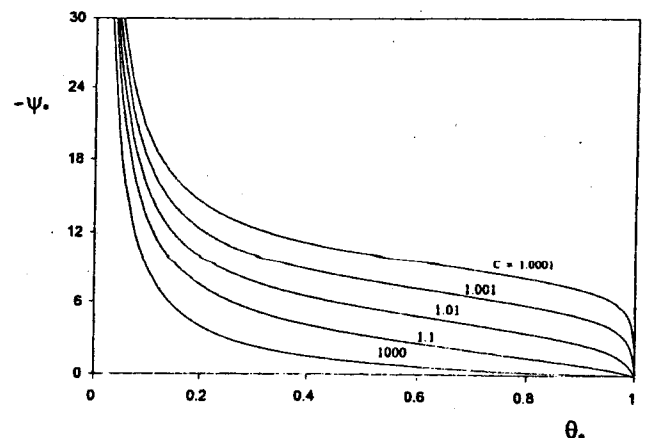


Fig. 2. Dimensionless matric potential ψ_* as a function of normalized water content θ_* .

was an adaptation of the Philip and Knight (1974) iterative procedure which does not provide exact parametric solution.

We assume that initially there is zero concentration of solute in the soil profile. A step of solute concentration c_0 is applied at the surface. By introducing the dimensionless variables $c_* = c/c_0$ and $E_* = E(t_s/\lambda_s^2)$ into equation (5), the advection-dispersion equation is expressed in the mathematical form:

$$\left(\theta_* + \frac{\theta_n}{\Delta\theta}\right) \frac{\partial c_*}{\partial t_*} = \frac{\partial}{\partial z_*} \left[\left(\theta_* + \frac{\theta_n}{\Delta\theta}\right) E_* \frac{\partial c_*}{\partial z_*} \right] - \left(q_* + \frac{K_n}{\Delta K}\right) \frac{\partial c_*}{\partial z_*} \quad (22)$$

The dimensionless initial and boundary conditions are written as:

$$c_* = 1, \quad z = 0, \quad t \geq 0 \quad (23)$$

$$c_* = 0, \quad t_* = 0, \quad z \geq 0 \quad (24)$$

The solution procedure employed to derive a parametric solution for equation (22) is similar to the one adopted by Wilson and Gelhar (1981). This solution procedure solves sequentially for the two mechanisms of solute transport, i.e., convection with the mean water velocity and dispersive mixing. The convection component is treated by the method of characteristics and the location of the solute front is determined. The dispersive component is then superimposed on the resulting front using the method of singular perturbation.

4-1. The Method of Characteristics

The convective component is obtained by setting the dispersive term in equation (22) equal to zero with the result:

$$\left(\frac{\theta_n}{\Delta\theta} + \theta_*\right) \frac{\partial c_*}{\partial t_*} + \left(q_* + \frac{K_n}{\Delta K}\right) \frac{\partial c_*}{\partial z_*} = 0 \quad (25)$$

In the method of characteristics, the partial differential equation is reduced to a set of ordinary differential equations known as the characteristic equations. Equation (25) is written in the mathematical form:

$$\left(\frac{dz_*}{dt_*}\right)_{c_*} = \frac{(K_n/\Delta K) + q_*}{(\theta_n/\Delta\theta) + \theta_*} \quad (26)$$

To obtain the characteristic equations, equation (26) is arranged in the form:

$$\left(\frac{\theta_n}{\Delta\theta} + \theta_*\right) dz_* - \left(\frac{K_n}{\Delta K} + q_*\right) dt_* = 0 \quad (27)$$

This last equation suggests the existence of an exact differential form provided the following equality holds:

$$\frac{\partial[(\theta_n/\Delta\theta) + \theta_*]}{\partial t_*} = -\frac{\partial[(K_n/\Delta K) + q_*]}{\partial z_*} \quad (28)$$

or equivalently,

$$\frac{\partial\theta_*}{\partial t_*} = -\frac{\partial q_*}{\partial z_*} \quad (29)$$

This is true by virtue of the continuity equation (4). Therefore, there exists a function η , such that

$$\frac{\partial\eta}{\partial z_*} = \frac{\theta_n}{\Delta\theta} + \theta_* \quad (30)$$

$$\frac{\partial\eta}{\partial t_*} = -\left(\frac{K_n}{\Delta K} + q_*\right) \quad (31)$$

Mathematically, it is obvious from equations (27), (30), and (31) that η is zero along the solute sharp front $z_*(t_*)$. The problem of determining the location of this sharp front is reduced to finding the analytical expression for η and setting it equal to zero.

Integrating equation (30) between the soil surface and an arbitrary depth z_* yields:

$$\eta = \int_0^{z_*} \left[\frac{\theta_n}{\Delta\theta} + \theta_*(z_*, t_*) \right] dz_* + f(t_*) \quad (32)$$

$f(t_*)$ is a sole function of time and is determined from equations (31) and (29) by introducing the flux boundary condition $q_{*,0}$ at the soil surface. Mathematically,

$$-\left[\frac{K_n}{\Delta K} + q_*(z_*, t_*) \right] = \frac{\partial\eta}{\partial t_*} = \int_0^{z_*} \frac{\partial\theta_*(z_*, t_*)}{\partial t_*} dz_* + \frac{df(t_*)}{dt_*} = -\int_0^{z_*} \frac{\partial q_*(z_*, t_*)}{\partial z_*} dz_* + \frac{df(t_*)}{dt_*} = q_{*,0} - q_*(z_*, t_*) + \frac{df(t_*)}{dt_*} \quad (33)$$

From equation (33) one has the result:

$$\frac{df(t_*)}{dt_*} = -\left(\frac{K_n}{\Delta K} + q_{*,0}\right) \quad (34)$$

Because the moisture flux at the soil surface is a step, the integration of equation (34) with respect to t_* yields the result,

$$f(t_*) = -\int_0^{t_*} \left(\frac{K_n}{\Delta K} + q_{*,0}\right) dt_* = -\left(\frac{K_n}{\Delta K} + q_{*,0}\right) t_* \quad (35)$$

The substitution of $f(t_*)$ into equation (32) yields the solution η for the characteristic:

$$\eta = \int_0^{z_*} \left[\frac{\theta_n}{\Delta\theta} + \theta_*(z_*, t_*) \right] dz_* - \left(\frac{K_n}{\Delta K} + q_{*,0}\right) t_* \quad \dots (36)$$

The first term on the right-hand side of the characteristic represents the cumulative change in water content between the soil surface and location z_* , whereas the second term is the cumulative volumetric water infiltrated at the surface between time zero and t_* . It is intuitively evident that those two quantities should be equal and that the location of the solute front is determined by setting $\eta = 0$. Physically, $\eta(z_*, t_*)$ could be thought of as a moving spatial coordinate in

relation to a solute front that was located at the soil surface at time zero. To determine η exactly, the parametric flow solutions of B&W are introduced at this stage. From the definition of the parameter ζ in equation (A-8), one has the relationship,

$$\int_0^{z_*} \theta_*(z_*, t_*) dz_* = Cz_* - \zeta \quad (37)$$

Thus we find that η is given by the expression:

$$\eta = \left(\frac{\theta_n}{C\Delta\theta} + 1 \right) Cz_* - \zeta - \left(\frac{K_n}{\Delta K} + q_{s,0} \right) t_* \quad (38)$$

which using the parametric solution in equation (21) for z_* , can be written in terms of ζ and the time τ as:

$$\eta = A_1 \tau + B_1 \zeta - C_1 \ln u(\zeta, \tau) \quad (39)$$

where A_1 , B_1 , and C_1 are constants defined symbolically as:

$$A_1 = C_1(\rho^2 + \rho) - \left(\frac{K_n}{m\Delta K} \right) \quad (40)$$

$$B_1 = C_1(2\rho + 1) - 1 \quad (41)$$

$$C_1 = 1 + \frac{\theta_n}{C\Delta\theta} \quad (42)$$

This solution for η is a parametric expression in ζ . The mathematical procedures to locate the position $z_f(t_*)$ of the sharp front include two steps: (1) set η equal to zero and determine ζ_f , and (2) set equation (38) equal to zero and solve for z_f in terms of dimensionless time t_* .

4-2. Perturbation Approximation of Dispersion (the Inner Expansion)

In the previous section, we determined the exact location $z_f(t_*)$ of the sharp front. The solute concentration $c_*(z_*, t_*)$ is one beyond this front and zero behind it. This contrast at the front boundary produces a singular problem that is tackled mathematically using the method of asymptotic expansion (e.g., Van Dyke, 1975; Dagan, 1971; Cole, 1968). For $E_* \ll 1$, an inner expansion valid in the vicinity of the front is introduced by stretching the coordinate system and rewriting the dispersion equation in the new coordinate system. An appropriate inner coordinate is:

$$x = \frac{(z_* - z_f)}{E_*} \quad (43)$$

By introducing the variable change in (43), the solute transport equation is then rewritten in the inner coordinate system as:

$$\frac{\partial}{\partial x} \left\{ \theta_*' \frac{\partial c_*}{\partial x} \right\} - (v - z_f') \theta_*' \frac{\partial c_*}{\partial x} = E_* \theta_*' \frac{\partial c_*}{\partial t_*} \quad (44)$$

where $v = (K_n/\Delta K + q_{s,0})/(\theta_n/\Delta\theta + \theta_*)$, and $z_f' = dz_f/dt_*$. In the following formulation, it is assumed that the variability of E_* and θ_* within the dispersive zone is negligible so that they can be replaced by their values at the front. Thus, after substituting the Taylor series expansion of θ_*' and v near the

sharp front into equation (44), and retaining the zeroth order terms in E_* (e.g., Dagan, 1971, pages 137 and 138), one obtains the closed form solution for c_* :

$$c_*(z_*, t_*) = \frac{1}{2} \operatorname{erfc} \left[\left(\frac{z_f'}{2E_* z_f'} \right)^{1/2} (z_* - z_f) \right] \quad (45)$$

with,

$$z_f = \frac{1}{CC_1} \left[\zeta_f + \left(\frac{K_n}{m\Delta K} + \rho \right) \tau \right] \quad (46)$$

$$z_f' = \frac{dz_f}{d\tau} = \frac{1}{CC_1} \zeta_f' + \left(\frac{K_n}{m\Delta K} + \rho \right) \quad (47)$$

$$z_f'' = \frac{d^2 z_f}{d\tau^2} = \frac{1}{CC_1} \zeta_f'' \quad (48)$$

$$\zeta_f' = \frac{C_1 u^{-1} u_{\zeta\zeta} / 4 - A_1}{B_1 - C_1 u^{-1} u_{\zeta}} \quad (49)$$

$$\zeta_f'' = C_1 u^{-2}$$

$$\frac{u[u_{\zeta\zeta}(\zeta_f')^2 + u_{\zeta\zeta\zeta}\zeta_f/4 + u_{\zeta\zeta\zeta\zeta}/16] - (u_{\zeta}\zeta_f' + u_{\zeta\zeta}/4)^2}{B_1 - C_1 u^{-1} u_{\zeta}} \quad (50)$$

and $u_{\zeta} = d^2 u/d\zeta^2$; $u_{\zeta\zeta} = d^3 u/d\zeta^3$; $u_{\zeta\zeta\zeta} = d^4 u/d\zeta^4$; and the

$$\frac{\partial^n u}{\partial \zeta^n} = 2^{n-1} \rho^n e^{-\zeta^2/\tau} (2e^{(\zeta+\rho\tau)^2} + (-1)^n (1 + \rho^{-1})^{n/2}$$

$$[f_1^- + (-1)^n f_1^+] - (-1)^n [f_2^- + (-1)^n f_2^+])$$

The limitation of the method of asymptotic expansion is well-documented in the literature on perturbation methods (Van Dyke, 1975; Cole, 1968). Similarly, the restrictions of this method to the application of solute transport models were investigated by several researchers (Dagan, 1971; Gelhar and Collins, 1971; Wilson and Gelhar, 1974). Wilson and Gelhar (1974) evaluated the restrictions on this assumption by introducing dimensionless variables and carrying perturbation expansion. Gelhar and Collins (1971) compared the solution based on the method of asymptotic expansion with numerical simulations. Their results indicated that the method of asymptotic expansion yields good results when the distance travelled by the solute is much larger than the dispersivity of the medium. They suggested that this condition is fulfilled in practical field applications. In the following section, the analytical solutions formulated here are compared with numerical simulations.

5. Features of the Solution

The solutions for water flow and solute transport in equations (20), (21), (38), (39), and (45) are analytic. So, the water content profile $\theta_*(z_*, t_*)$ and the concentration of solute in the soil $c_*(z_*, t_*)$ are determined without iterative steps commonly used in numerical schemes. The input requirement for an analytical simulation includes (1) soil hydraulic properties: the saturated conductivity K_s , the saturated water content θ_s , the parameter C , and the capillary length scale λ_s ; (2) the soil initial moisture content θ_n and the water flux boundary condition $q_{s,0}$; and (3) the dispersion

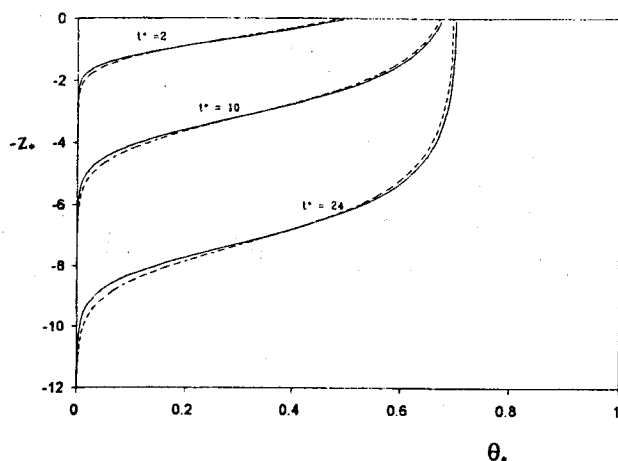


Fig. 3. Analytical (solid line) and numerical (dashed line) solutions of θ_* .

coefficient E_* . For the sake of illustration, we choose $C = 1.2$, $\lambda_s = 3.84$ cm, $K_s = 4.5$ cm.hr⁻¹, $q_{*,0} = 0.2$, $\theta_s = 0.4$, $\theta_n = 0.08$, and $E_* = 10^{-3}$. The values of these parameters reflected the conditions at a site in Golden, Colorado. The sorptive time scale t_s is calculated from equation (15) to be 0.27 hour. Numerical simulations with the PRINCETON finite-element model (Celia, 1991; Celia et al., 1990) were conducted for the same input conditions. The time step in the PRINCETON code is adjusted internally depending on the number of nonlinear iterations (Celia, 1991). For the finite-element numerical scheme, we chose an element size of one-fourth the capillary length scale (i.e., $\lambda_s/4$) or 0.96 cm for this particular case. Figure 3 shows a good match between the analytical and numerical solutions for the normalized water content θ_* at t_* equal 2, 10, and 24 at this level of discretization. However, when solving the advection-dispersion equation, spatial oscillations (overshoot and undershoot) and numerical dispersion were significantly large for the conditions of our simulation. Figure 4 compares the analytical and numerical solutions when we reduced the mesh grid in the finite-element code to one-eighth the capillary length

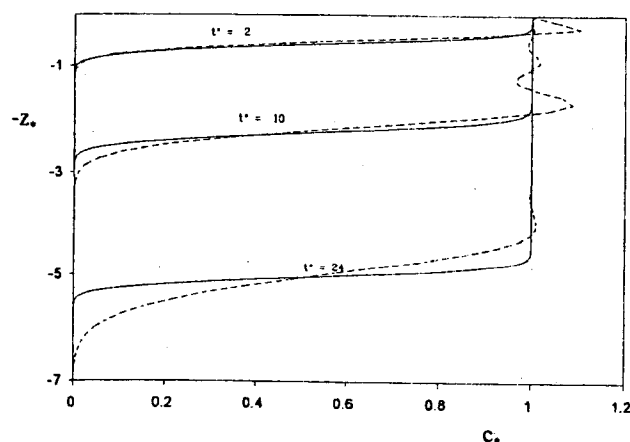


Fig. 4. Analytical (solid line) and numerical (dashed line) solutions of c_* .

scale. While the match between the concentrations is not very satisfactory, especially at $t_* = 24$, the trade-off between numerical dispersion and spatial oscillations is typical for advection-dominated solute transport like the case considered here. Indeed, Huyakorn and Pinder (1983, page 207) cautioned that "Because advection-dominated transport problems are difficult to solve numerically, the user of computer models needs to test the adequacy of selected mesh and time step discretizations to avoid obtaining misleading or meaningless numerical solutions. For this reason we recommend the use of available analytical solutions for calibration"

Conclusion

We extended the Broadbridge and White infiltration model to derive a closed form analytical solution for the transport of a conservative solute in the unsaturated zone. The presented analytical solutions are exact and parametric, and the soil-water content and the solute concentration solutions are easily generated and programmed. The computation time required to achieve a complete solution is insignificant when compared with iterative numerical schemes. The numerical problems involved in solving advection-dominated transport are well recognized among practitioners. The developed analytical solutions should provide a mean for calibrating and evaluating the accuracy of existing numerical codes.

Appendix

The expressions appearing in the θ_* solution in equations (20) and (21) have the form (see also B&W, 1988):

$$u = \frac{1}{2} e^{-\zeta^2/\tau} [2e^{(\zeta+\rho\tau)^2/\tau} + f_1^- + f_1^+ - (f_2^- + f_2^+)] \quad (\text{A-1})$$

where,

$$f_1^{\pm} = f[(\zeta \pm \rho(1 + \rho^{-1})^{1/2}\tau)/\tau^{1/2}] \quad (\text{A-2})$$

$$f_2^{\pm} = f[(\zeta \pm \rho\tau)/\tau^{1/2}] \quad (\text{A-3})$$

$$f(x) = e^{x^2} \text{erfc}(x) \quad (\text{A-4})$$

and,

$$\rho = q_{*,0}/m \quad (\text{A-5})$$

$$m = 4C(C - 1) \quad (\text{A-6})$$

$$\tau = mt_* \quad (\text{A-7})$$

$$\zeta = \int_0^{z_*} (C - \theta_*) dz_* \quad (\text{A-8})$$

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